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STABILITY FUNCTIONS FOR STRUCTURAL MASONRY

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Abstract—An analytical method for the caluclation of the stiffness and carry-over coefficients for masonry beam-columns is presented. The masonry element is made of a material which can resist little or no tension. It is shown that the stability functions for the conventional frame element represent an upper or lower bound for the functions discussed here; further, that unlike in the conventional case, specific knowledge of the eccentricity ratio is required for a unique solution.

NOTATION

b width of column
c carry-over factor of a beam subject to an axial force
C constant of integration
d depth of column
e eccentricity at top of pier
h length of basic column
L Σl
m $6e/d$
p deflection parameter
P axial load
P_E $\pi^2 EI/L^2$
r^2 $(3-\alpha)/(1+2p-\alpha)$
$r_{p\alpha}$ $4/r^2(1-p)+a^2$
s non-dimensional stiffness of a beam subject to an axial force
x, u coordinates
z^2 $(3-m)/(1+2p-m)$
z_1^2 $(3-m_1)/(1+2p-m_1)$
z_p $4/z^2(1-p)$
α $1+|\sigma_t/\sigma_{av}|$
λ $\sqrt{(P/EI)}$
$\sigma, \sigma_t, \sigma_{av}$ compressive stress, tensile resistance, P/bd.

1. INTRODUCTION

Analytical solutions of rigid frames fall, broadly, in two basic groupings; those resting on the flexibility (displacement) method, and those which are derived by applying the stiffness, or force, method. The latter, of which the moment-distribution method is an example, requires stiffness coefficients at both ends of the member, or one stiffness coefficient at the rotating end and a carry-over coefficient at the fixed end. For a homogeneous, prismatic member in the absence of an axial load these values are $4EI/l$ and 0.5, respectively, but they change in the presence of an axial load [1], Fig. 1. The concept of a homogeneous member will be discarded in this discussion; instead, the emphasis will be placed on a member of rectangular cross section made of a material which can resist little or no tension.

If the beam-column is a brick pier of low quality mortar it will have practically zero tensile resistance; in such a column cracking will occur at the slightest tendency for tension. The effective depth of the section decreases after cracking occurs and because of this the beam-column may be looked at as a bar with varying moment of inertia. This variation is, however, unknown because it depends on the deflected shape; furthermore, it is not symmetrically placed about the material axis (Fig. 2). If $\sigma_t = 0$, stress distributions are trapezoidal in the uncracked part and triangular in the cracked zone. For a material with a small amount of tensile strength such tension will build up on the convex face and a skew trapezoidal stress distribution will balance the external forces in the uncracked portion. When this tensile limit is exceeded triangular distributions will appear over part of the depth of the column in the cracked zone (Fig. 2).

Columns made of no-tension material have been considered by Angervo [2], Chapman and

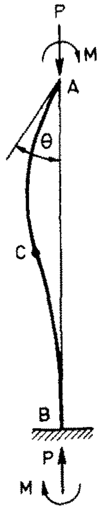

Fig. 1.

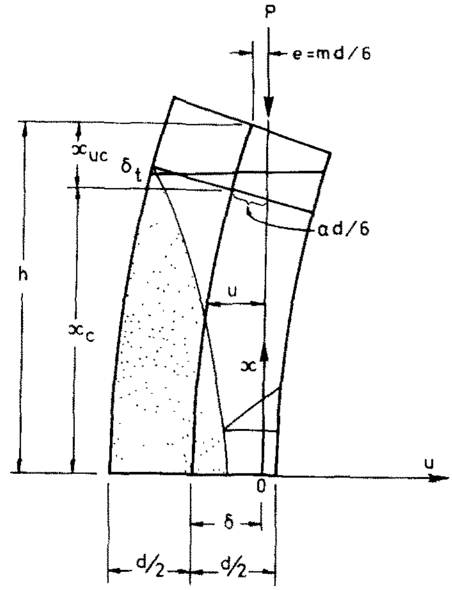

Fig. 2.

Fig. 1. Basic column.
Fig. 2. Deflected shape of pier.

Slatford[3], Yokel[4], Risager[5] and Frisch-Fay[6]. Portal frames have been the subject of investigation by Sahlin and Hellers[7].

2. FORMULATION OF THE PROBLEM AND THE BASIC CASES

The basic column to be analised is of a rectangular cross section (Fig. 2), having a width b. A load P is placed eccentrically along the minor axis at the free end. It is made of a material that can resist tension $\sigma_t \geq 0$.

The differential equation governing the uncracked part of the column in Fig. 2 is

$$\frac{d^2u}{dx^2} + u/\lambda^2 = 0$$

while the cracked zone is controlled by

$$\frac{d^2u}{dx^2} + \frac{2P}{9Eb(0.5d-u)^2} = 0. \qquad [6]$$

Depending on the eccentricity of P we distinguish between columns with no cracked zone, columns with cracked zones only, and columns which contain both.

Case A. If the line of thrust is wholly within the permissible kerns, all sections are uncracked. This will be the case if $e \leq \alpha d/6$, and, in addition, $\delta \leq \alpha d/6$.

Direct integration of

$$\frac{d^2u}{dx^2} + u/\lambda^2 = 0 \tag{1}$$

leads to

$$\frac{du}{dx} = \frac{1}{\lambda}\sqrt{(\delta^2 - u^2)} = \frac{1}{\lambda}\sqrt{\left(\frac{d^2}{36} + (p+p^2)\frac{d^2}{9} - u^2\right)} \tag{2}$$

after adjusting the integration constant to satisfy

$$\left.\frac{du}{dx}\right|_{\substack{x=0\\u=\delta}} = 0,$$

and to

$$\left.\frac{du}{dx}\right|_{u=e}=\frac{d}{6\lambda}\sqrt{[(1+2p)^2-m^2]} \tag{3}$$

where $\delta = d/6+pd/3$ and $m=6e/d$. Further integration of eqn (2) results in

$$\frac{h}{\lambda}=0.5\pi-\sin^{-1}\frac{m}{1+2p} \tag{3a}$$

because the integration satisfies $u(0)=\delta$, and $e/\delta = m/(1+p)$. Case A is available for $0\leqslant m\leqslant \alpha$ and $(m-1)/2\leqslant p\leqslant(\alpha-1)/2$.

Case B. Here the line of thrust lies partially within the permissible kern which in turn depends on α. The lower zone is cracked (Fig. 2). The differential equation applying to the lower portion is

$$\frac{d^2u}{dx^2}+\frac{2P}{9Eb(0.5d-u)^2}=0 \tag{4}$$

and direct integration gives the slope in the cracked (lower) region as

$$\frac{du}{dx}=\frac{d}{\lambda}\sqrt{\left(\frac{\delta-u}{(1-p)(0.5d-u)}\right)} \tag{5}$$

where the integration constant satisfies

$$\left.\frac{du}{dx}\right|_{\substack{x=0\\ u=\delta}}=0.$$

For the uncracked upper part eqn (1) obtains. Integration results in

$$\frac{du}{dx}=\sqrt{(C-u^2/\lambda^2)}\,. \tag{6}$$

The slopes of eqns (5) and (6) are identical where cracked and uncracked portions join. For a no-tension pier this will occur at $u=d/6$ ($\alpha=1$); if, however, the convex face can resist σ_t the eqns (5) and (6) are identical at $u=\alpha d/6$. It can be shown that the shape in the uncracked (upper) part is

$$\frac{du}{dx}=\frac{1}{\lambda}\sqrt{\left(\frac{d^2}{9}\left(\frac{1+2p-\alpha}{(1-p)(3-\alpha)}+\frac{\alpha^2}{4}\right)-u^2\right)} \tag{7}$$

and the end slope is

$$\left.\frac{du}{dx}\right|_{u=e}=\frac{d}{3\lambda}\sqrt{\left(\frac{1+2p-\alpha}{(1-p)(3-\alpha)}+\frac{\alpha^2-m^2}{4}\right)}. \tag{8}$$

Direct quadrature of eqn (5) leads to the length of the cracked section

$$x_c=2\lambda(1-p)^{3/2}\int_{r(\alpha d/6)}^{\infty}\frac{r^2}{(r^2-1)^2}\,ds \tag{9}$$

where a new variable $r(u)=\sqrt{[(0.5d-u)/(\sigma-u)]}$ has been introduced. The length of the cracked zone is now

$$x_c=0.5\lambda(1-p)^{3/2}\left(\ln\frac{r+1}{r-1}+\frac{2r}{r^2-1}\right) \tag{10}$$

where $r=r(\alpha d/6)=\sqrt{[(3-\alpha)/(1+2p-\alpha)]}$.

In order to obtain the length of the uncracked portion integrate eqn (7) from $u = \alpha d/6$ to $u = e$. Remembering that for a positive slope $\mathrm{d}u/\mathrm{d}x$, u is negative,

$$x_{uc} = \lambda \int_{u=e}^{u=\alpha d/6} \frac{\mathrm{d}u}{\sqrt{\left(\frac{d^2}{9}\left(\frac{1+2p-\alpha}{(1-p)(3-\alpha)}+\frac{\alpha^2}{4}\right)-u^2\right)}} = \lambda\left[\sin^{-1}\frac{3u}{d\sqrt{\left(\frac{1}{(1-p)r^2}+\frac{\alpha^2}{4}\right)}}\right]_{u=e}^{u=\alpha d/6}$$

$$= \lambda \sin^{-1}\frac{\alpha/2}{\sqrt{\left(\frac{1}{(1-p)r^2}+\frac{\alpha^2}{4}\right)}} - \lambda \sin^{-1}\frac{m/2}{\sqrt{\left(\frac{1}{(1-p)r^2}+\frac{\alpha^2}{4}\right)}} \tag{11}$$

Adding eqns (10) and (11)

$$\frac{h}{\lambda} = 0.5(1-p)^{3/2}\left[\ln\frac{r+1}{r-1}+\frac{2r}{r^2-1}\right] + \sin^{-1}\frac{\alpha/2}{\sqrt{\left(\frac{1}{(1-p)r^2}+\frac{\alpha^2}{4}\right)}} - \sin^{-1}\frac{m/2}{\sqrt{\left(\frac{1}{(1-p)r^2}+\frac{\alpha^2}{4}\right)}}. \tag{12}$$

The results are valid for $m < \alpha$, and $(\alpha-1)/2 \leq p < 1$.

Case C. The line of thrust now lies wholly outside the permissible kern; all sections are cracked.

Using eqns (4) and (5) we find that the end slope

$$\left.\frac{\mathrm{d}u}{\mathrm{d}x}\right|_{u=e} = \frac{d}{3\lambda}\sqrt{\left(\frac{1+2p-m}{(1-p)(3-m)}\right)}. \tag{13}$$

The length of the cracked portion now equals the length of the pier h. From eqn (9)

$$h = 0.5\lambda(1-p)^{3/2}\left[\ln\frac{r+1}{r-1}+\frac{2r}{r^2-1}\right]_{u=e,r=r(m)}^{u=\delta,r=\infty} \tag{14}$$

where $r(m) = \sqrt{[(3-m)/(1+2p-m)]} = z$ or

$$\frac{h}{\lambda} = 0.5(1-p)^{3/2}\left(\ln\frac{z+1}{z-1}+\frac{2z}{z^2-1}\right). \tag{15}$$

Equation (15) is valid for $\alpha \leq m < 3$, $0.5(m-1) < p \leq 1$.

3. BASIC EQUATIONS

The original beam column, fixed at B and rotated through θ at A, subject to an axial load P and having no translation between A and B, is shown in Fig. 1. On replacing M by Pe we can substitute the forces and moments in Fig. 1 by those shown in Fig. 3; for a fixed e_1 the other eccentricity e_2 must be selected such that

$$\psi(\Sigma l) = e_1 + e_2 \tag{16}$$

where Σl is the aggregate of the lengths that make up AB and ψ is the slope at B. The stiffness factor

$$s = \frac{Pe_1}{\frac{EI}{\Sigma l}\theta} \tag{17}$$

and the carry-over factor

$$c = m_2/m_1$$

follow immediately.

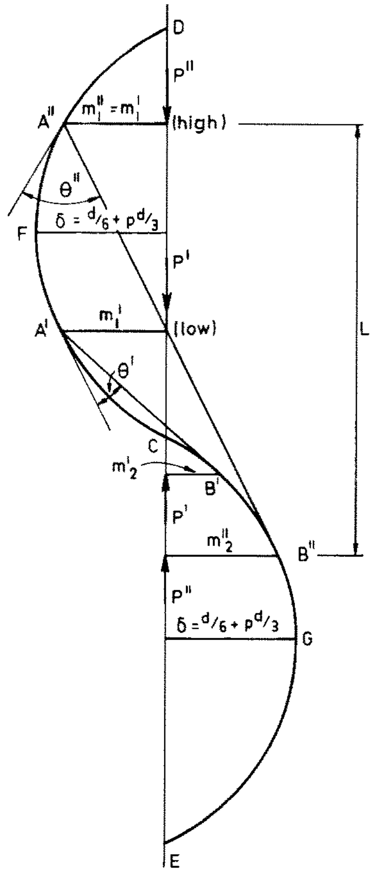

Fig 3. Compositions of basic cases.

We distinguish two positions for the same $m_1 = 6e_1/d$; the low position, making a pier $A'B'$ with eccentricities m_1' and m_2'; and the high position giving a strut $A''B''$ and eccentricities m_1'', and m_2'' (Fig. 3).

Let the slopes at B' and B'' be ψ' and ψ'', respectively. Then,

$$\theta' = \psi' - \left.\frac{du}{dx}\right|_{e=m_1'}$$

and

$$\theta'' = \psi'' + \left.\frac{du}{dx}\right|_{e=m_1''} . \tag{19}$$

Since both branches have the same slope at C it follows that the deflection parameter p is also identical for both branches, that is, for DFC, and CGE.

For a fixed m_1 and p the aggregate length $\Sigma l = L$ can be calculated. Then,

$$P/P_E = A^2/\pi^2 \tag{20}$$

where $A = L/\lambda$, and $P_E = EI\pi^2/L^2$.

Equation (16) is the basic equation to be solved; the unknown is m_2. There is, however, a variety of combinations, depending on whether m_1 is inside or outside the effective kern, whether it attracts m_2 inside or outside the permissible kern, whether for $m_1 < \alpha$ the line of thrust is wholly within the kern, wholly outside, or a combination of these and, finally, depending on whether m_1 is in high or low position (Fig. 3). These possibilities, leading to 10 different cases, will be considered in the Appendix.

4. NUMERICAL RESULTS

A computer program will scan m_1 from 0.1 to 2.6 searching for an appropriate m_2, for six values of p placed within the range applicable. The computation begins with A_1 (eqn (21) in the Appendix); A_2 and A_3 follow. Here it branches to A_4 or A_6. Next it takes one of three possible routes: $A_7 - A_9$, or $A_7 - A_{10}$, or $A_8 - A_{10}$.

The stiffness and carry-over values for a certain ratio of P/P_E depend, unlike for conventional columns, on the eccentricity of the load at the rotated end. For comparison, s and c are shown in Fig. 4[1]. The dual values of s and c corresponding to any particular value of P/P_E are a feature not found in the conventional stability functions and call for a brief comment. The stiffness values begin on the upper, descending branch (full line) and terminate at the vertical tangent; the c values, on the other hand, are on the lower,ascending branch (full line). Also shown on Fig. 4 is the relationship of s and c for the conventional and cracked column.

For $\alpha = 1, 1.5, 2$, a family of s and c curves is presented (Figs. 5–13). The last two values of α refer to slight, and moderate tensile strength, respectively, while $\alpha = 1$ means a no-tension material. Figures 6, 9 and 12 show carry-over factors vs P/P_E; for low values of m(0.1–1.0) Figs. 7, 10 and 13 have been provided.

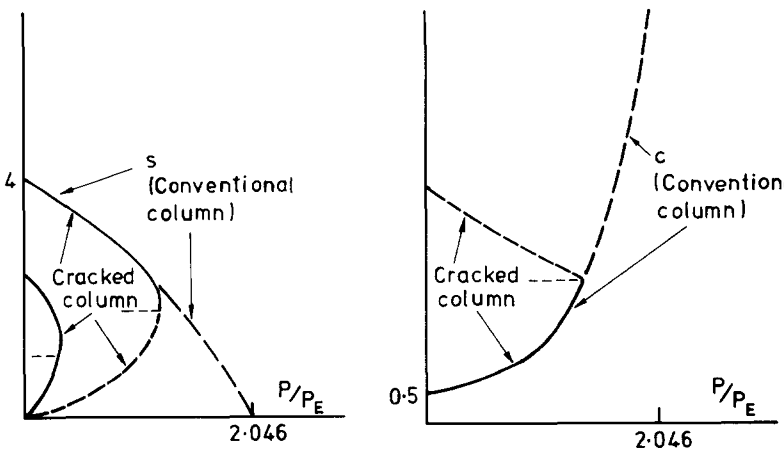

Fig. 4. Comparison between conventional and masonry column.

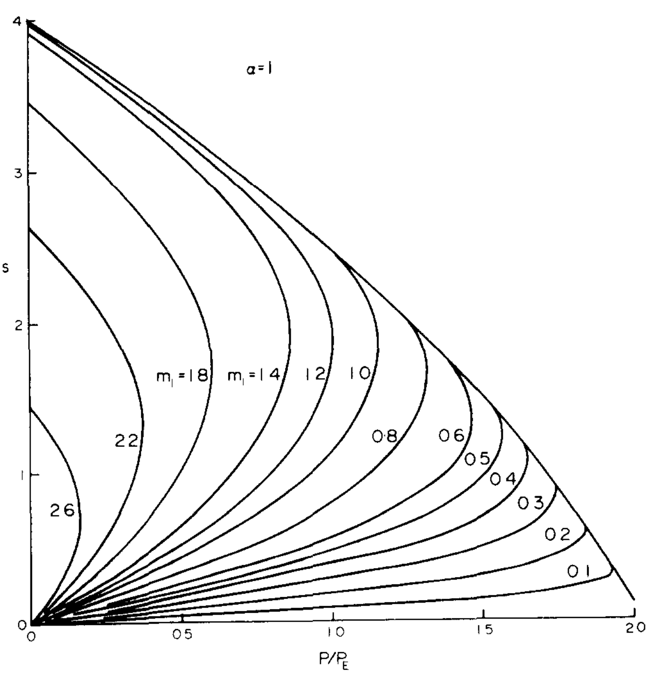

Fig. 5 Stiffness coefficients for $\alpha = 1$.

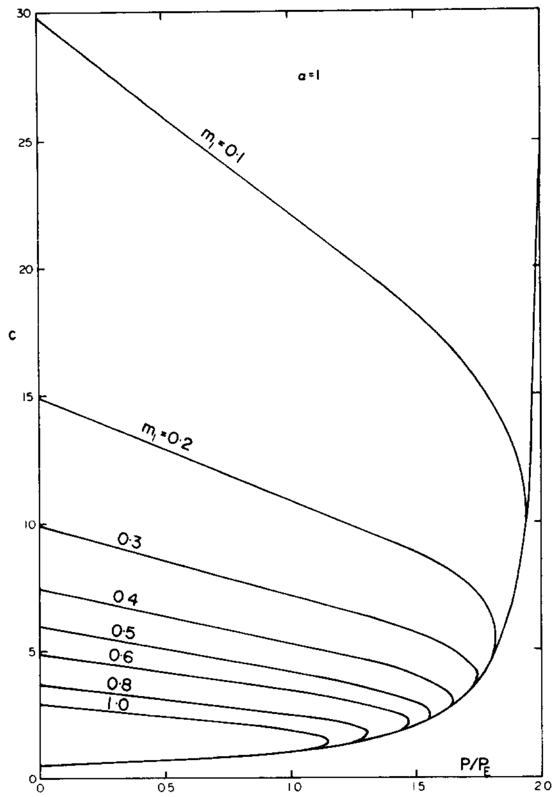

Fig. 6. Carry-over factors for $\alpha = 1$.

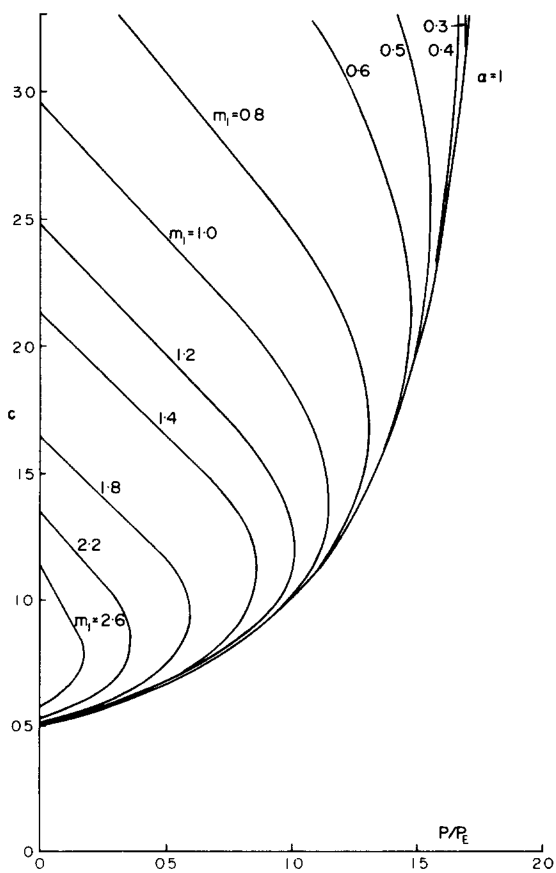

Fig. 7. Carry-over factors for $\alpha = 1$.

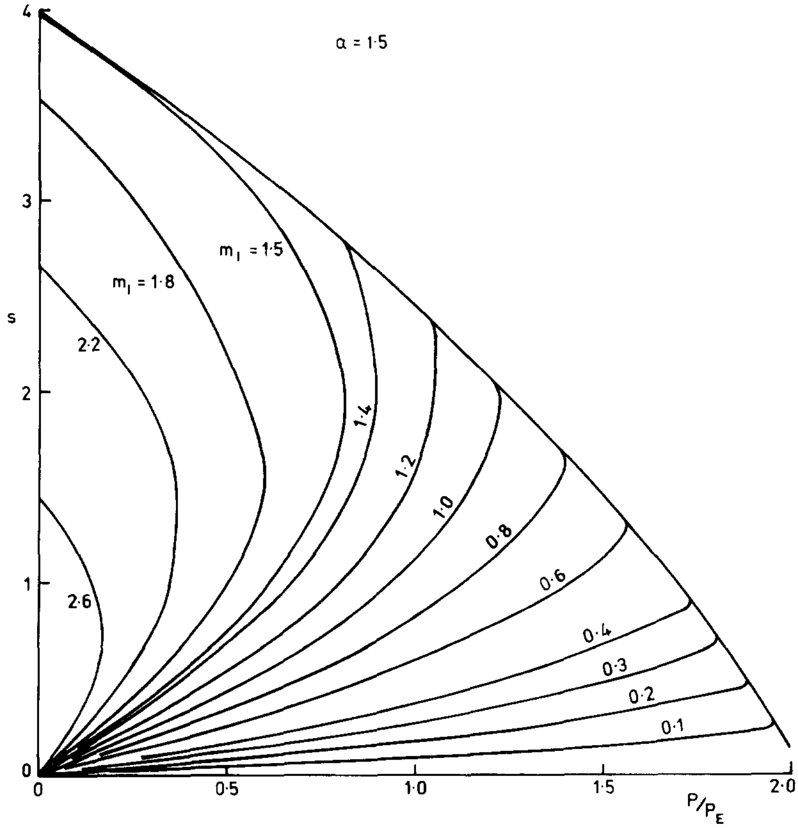

Fig. 8. Stiffness coefficients for $\alpha = 1.5$.

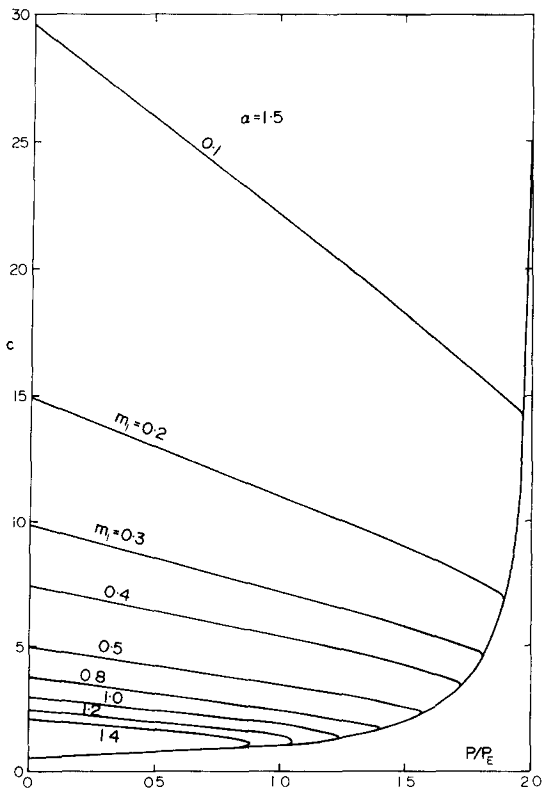

Fig. 9. Carry-over factors for $\alpha = 1.5$.

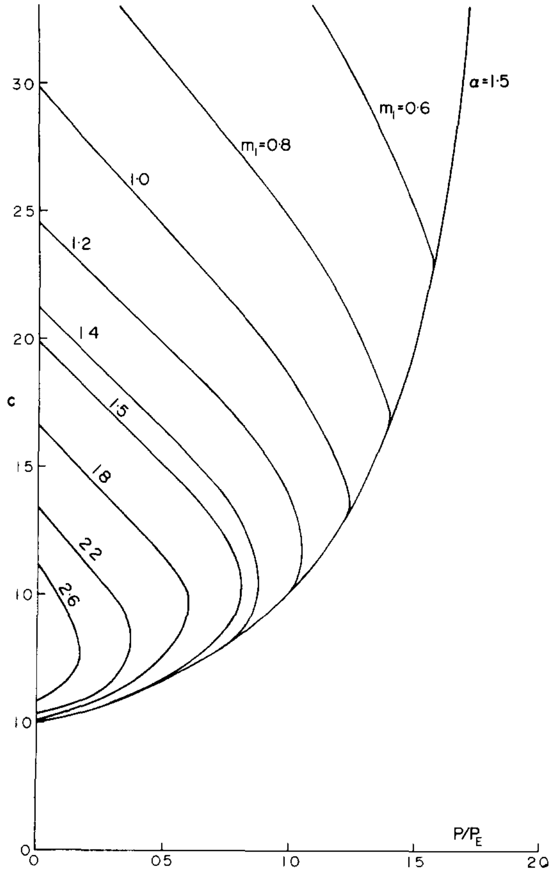

Fig. 10. Carry-over factors for $\alpha = 1.5$.

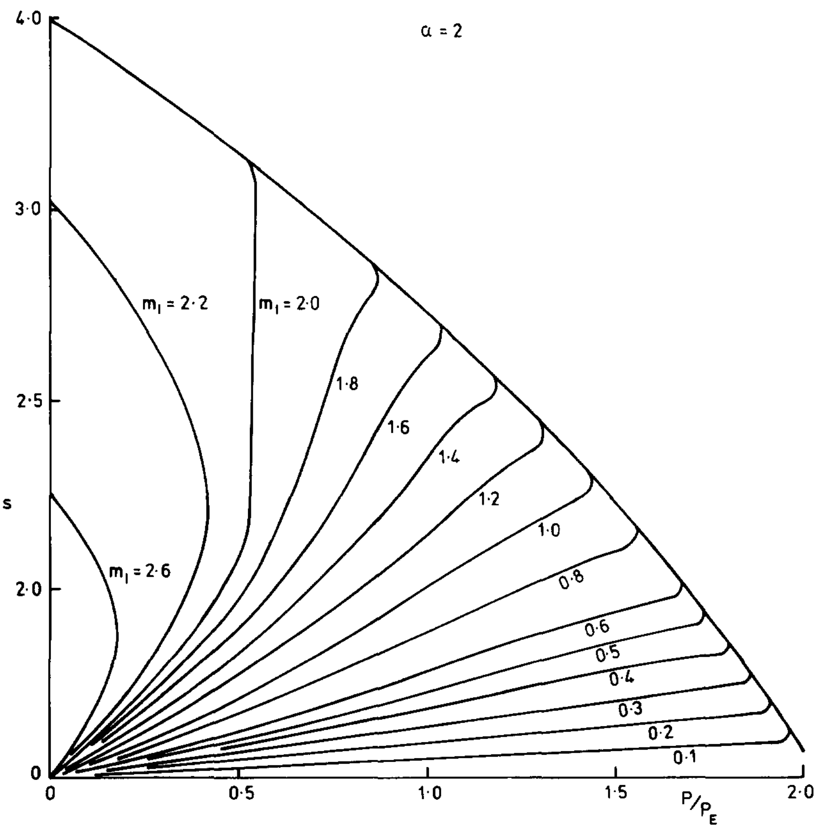

Fig. 11. Stiffness coefficients for $\alpha = 2$.

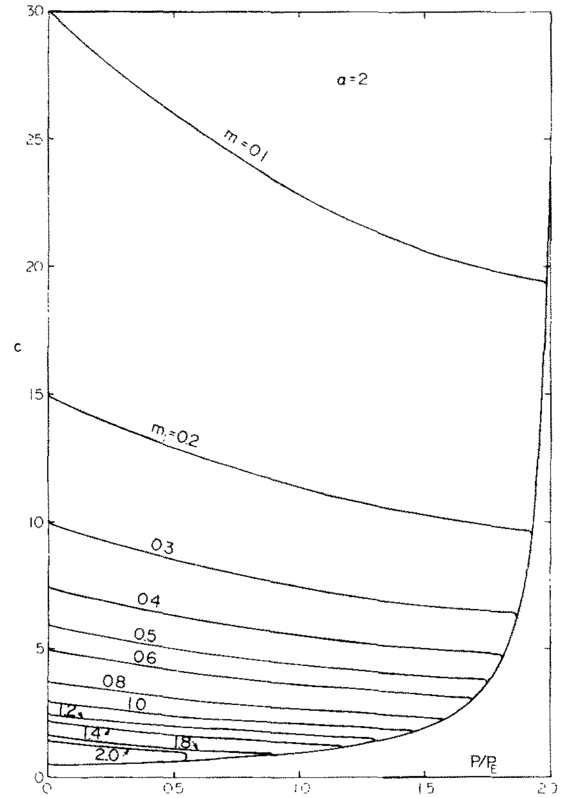

Fig. 12. Carry-over factors for $\alpha = 2$.

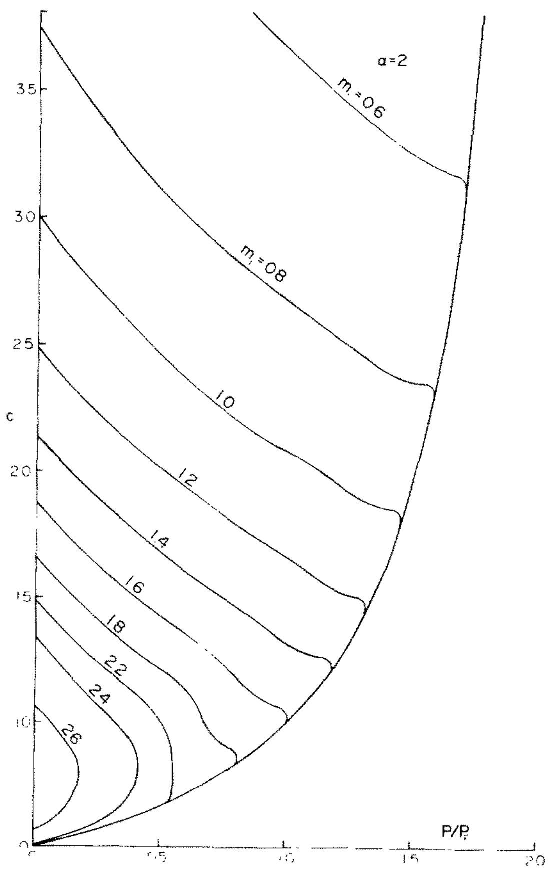

Fig. 13. Carry-over factors for $\alpha = 2$.

5. EXAMPLE

Consider a steel girder 6 m long fixed at one end, supported by a brick pier at the other, and carrying 40 kN/m distributed load. Let $E = 200{,}000$ MPa and $I = 375 \times 10^6$ mm^4 for the girder; the brick pier is 12 m high, and its dimensions are $d = 0.2$ m, $b = 0.34$ m. Let $E = 20{,}000$ MPa for the pier.

If the girder acts as simply supported on the pier the reaction is 90 kN and the rotation $\varphi' = (1/48)\, wL^3/EI = 0.00245$. Assume that $P = 90$ kN acts at $e = 0.060$ m on the pier, thus $m = 1.8$. With $P/P_E = 0.286$, $s_{\alpha=1} = 2.9$ and the rotation of the top of the pier $\varphi'' = 0.00247$. The assumed eccentricity is, therefore, almost correct. From $c_{\alpha=1} = 0.6$, the eccentricity at the base is 0.036 m. For a conventional column $s = 3.6$ and $c = 0.58$; a rigid connection results in $e = 0.03$ m.

6. CONCLUSION

By applying a non-linear differential equation, in addition to a linear one, stiffness and carry-over factors can be obtained for a column made of no-tension material ($\alpha = 1$) or of a material resisting little tension ($\alpha = 1.5 - 2$). The basic operational tools applicable to conventional structures can thus be extended to masonry structures.

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APPENDIX

Case 1. ($m_1 < \alpha$, $m_2 < \alpha$, $0.5(m-1) < p < 0.5(\alpha - 1)$), m_1 in low position. In Case 1 the line of thrust lies wholly within the permissible kern and σ_t is nowhere exceeded on the convex face. Appropriate lengths are obtained from eqn (3a) and Fig. 3

$$CF/\lambda = 0.5\pi, \quad CA/\lambda = \sin^{-1}\frac{m_1}{1+2p}, \quad CB'/\lambda = \sin^{-1}\frac{m_2'}{1+2p}.$$

With eqn (3) for the slope at B' and eqn (19) for θ we have

$$\frac{m_1 + m_2'}{\sqrt{[(1+2p)^2 - m_2'^2]}} = \sin^{-1}\frac{m_1}{1+2p} + \sin^{-1}\frac{m_2'}{1+2p} \tag{20}$$

for the solution of m_2'. Let the RHS of eqn (20) be A_1 when m_2' satisfies that equation; then

$$P/P_E = A_1^2/\pi^2, \quad s_1 = \frac{A_1 m_1}{\sqrt{((1+2p)^2 - (m_2')^2)} - \sqrt{((1+2p)^2 - m_1^2)}} \tag{21}$$

$$c_1 = m_2'/m_1.$$

Case 2. ($m_1 < \alpha$, $m_2 < \alpha$, $0.5(m-1) < p < 0.5(\alpha - 1)$), m_1 in high position. Again, the line of thrust lies wholly within the permissible kern and σ_t is nowhere exceeded on the convex face. From Fig. 3

$$A''B''/\lambda = \pi - \sin^{-1}\frac{m_1}{1+2p} + \sin^{-1}\frac{m_2''}{1+2p},$$

and m_2'' is found by solving

$$\frac{m_1 + m_2''}{\sqrt{((1+2p)^2 - (m_2'')^2)}} = \pi - \sin^{-1}\frac{m_1}{1+2p} + \sin^{-1}\frac{m_2''}{1+2p}. \tag{22}$$

Following the same approach as in Case 1, there is

$$P/P_E = A_2^2/\pi^2, \quad s_2 = \frac{A_2 m_1}{\sqrt{((1+2p)^2 - (m_2'')^2)} + \sqrt{((1+2p)^2 - m_1^2)}}, \quad c_2 = m_2''/m_1. \tag{23}$$

Results of Case 1 and 2 are identical with the stability values of conventional, tension resisiting columns (beam-columns) because tension in these two cases will everywhere be below σ_t, hence all sections are uncracked.

Case 3. ($m_1 < \alpha, m_2 < \alpha, 0.5(\alpha - 1) \leq p \leq 1$), m_1 in low position. Since the line of thrust lies everywhere inside the effective kern this case is identical (with the exception of the range for p) with Case 1.

Case 4. ($m_1 < \alpha, m_2 < \alpha, 0.5(\alpha - 1) < p < 1$), m_1 in high position. The column now is $A''CB''$ and tension in excess of σ_t may develop in the region about F leading to cracked sections in that area.

$$(A''CB'')/\lambda = (1-p)^{3/2}\left[\ln\frac{r+1}{r-1} + \frac{2r}{r^2-1}\right] + 2\sin^{-1}\frac{\alpha}{\sqrt{(r_{p\alpha})}} - \sin^{-1}\frac{m_1}{\sqrt{(r_{p\alpha})}} + \sin^{-1}\frac{m_2''}{\sqrt{(r_{p\alpha})}} = A_4 \tag{24}$$

where

$$r_{p\alpha} = \frac{4}{r^2(1-p)} + \alpha^2$$

and

$$r^2 = \frac{3-\alpha}{1+2p-\alpha}.$$

m_2'' is found from

$$\frac{m_1 + m_2''}{\sqrt{(r_{p\alpha} - (m_2'')^2)}} = A_4 \tag{25}$$

followed by

$$P/P_E = A_4^2/\pi^2, \quad s_4 = \frac{A_4 m_1}{\sqrt{(r_{p\alpha} - (m_2'')^2)} + \sqrt{(r_{p\alpha} - m_1^2)}}, \quad c_4 = m_2''/m_1. \tag{26}$$

Case 5. ($m_1 < \alpha, m_2 > \alpha, 0.5(\alpha - 1) < p < 1$), m_1 in low position. This case is not possible because $m_1 > m_2'$ (Fig. 3).

Case 6. ($m_1 < \alpha, m_2 > \alpha, 0.5(\alpha - 1) < p < 1, 0.5(m_2 - 1) < p < 1$), m_1 in high position. From Fig. 3, and with eqns (12) and (15),

$$A''B''/\lambda = \frac{3}{2}(1-p)^{3/2}\left[\ln\frac{r+1}{r-1} + \frac{2r}{r^2-1}\right] + 3\sin^{-1}\frac{\alpha}{\sqrt{(r_{p\alpha})}} - \sin^{-1}\frac{m_1}{\sqrt{(r_{p\alpha})}} - 0.5(1-p)^{3/2}\left[\ln\frac{z+1}{z-1} + \frac{2z}{z^2-1}\right] = A_6 \tag{27}$$

where

$$z^2 = \frac{3 - m_2''}{1 + 2p - m_2''}.$$

m_2'' may be solved from

$$\frac{m_1 + m_2''}{\sqrt{(z_p)}} = A_6 \tag{28}$$

and then, from eqns (8) and (13)

$$P/P_E = A_6^2/\pi^2, \quad s_6 = \frac{A_6 m_1}{\sqrt{(z_p)} + \sqrt{((r_{p\alpha} - m_1^2))}}, \quad c_6 = m_2''/m_1 \tag{29}$$

where

$$z_p = \frac{4}{z^2(1-p)}.$$

Case 7. ($m_1 > \alpha, m_2 < \alpha, 0.5(m_1 - 1) < p < 1$), m_1 in low position. From Fig. 3, and with eqns (12) and (15)

$$A'B'/\lambda = 0.5(1-p)^{3/2}\left[\ln\frac{r+1}{r-1} + \frac{2r}{r^2-1}\right] + \sin^{-1}\frac{\alpha}{\sqrt{(r_{p\alpha})}} - 0.5(1-p)^{3/2}\left[\ln\frac{z_1+1}{z_1-1} + \frac{2z_1}{z_1^2-1}\right] + \sin^{-1}\frac{m_2'}{\sqrt{(r_{p\alpha})}} = A_7 \tag{30}$$

where

$$z_1^2 = \frac{3 - m_1}{1 + 2p - m_1}.$$

Solution of

$$\frac{m_1 + m_2'}{\sqrt{(r_{p\alpha} - (m_2')^2)}} = A_7 \tag{31}$$

yields m_2', while

$$P/P_E = A_7^2/\pi^2, \quad s_7 = \frac{A_7 m_1}{\sqrt{(r_{p\alpha} - (m_2')^2)} - \sqrt{\left(\frac{4}{z_1^2(1-p)}\right)}}, \quad c_7 = m_2'/m_1 \tag{32}$$

Case 8. $(m_1 > \alpha, m_2 > \alpha, 0.5(m_1 - 1) < p < 1, m_1 > m_2)$, m_1 in low position. Here, as seen from Fig. 3,

$$A'B'/\lambda = (1-p)^{3/2}\left[\ln\frac{r+1}{r-1}+\frac{2r}{r^2-1}\right]+2\sin^{-1}\frac{\alpha}{\sqrt{(r_{p\alpha})}}-0.5(1-p)^{3/2}\left[\ln\frac{z_1+1}{z_1-1}+\frac{2z_1}{z_1^2-1}\right]$$
$$-0.5(1-p)^{3/2}\left[\ln\frac{z_2+1}{z_2-1}+\frac{2z_2}{z_2^2-1}\right]=A_8$$

because $A'B' = CF + CG - A'F - B'G$; also,

$$z_2^2 = \frac{3-m_2}{1+2p-m_2}$$

m_2' is obtained by solving

$$\frac{m_1+m_2'}{\sqrt{(z_p)}} = A_8 \tag{34}$$

Further,

$$P/P_E = A_8^2/\pi^2,\quad s_8 = \frac{A_8 m_1}{\sqrt{(z_p)}-\sqrt{\left(\frac{4}{z_1^2(1-p)}\right)}},\quad c_8 = m_2'/m_1. \tag{35}$$

Case 9. $(m_1 > \alpha, m_2 < \alpha, 0.5(m_1 - 1) < p < 1, 0.5(m_2 - 1) < p < 1)$, m_1 in high position. This is the high position corresponding to Case 7.

$$A''B''/\lambda = 0.5(1-p)^{3/2}\left[\ln\frac{r+1}{r-1}+\frac{2r}{r^2-1}\right]+\sin^{-1}\frac{\alpha}{\sqrt{(r_{p\alpha})}}+\sin^{-1}\frac{m_2''}{\sqrt{(r_{p\alpha})}}+0.5(1-p)^{3/2}\left[\ln\frac{z_1+1}{z_1-1}+\frac{2z_1}{z_1^2-1}\right]=A_9. \tag{36}$$

Upon solving

$$\frac{m_1+m_2''}{\sqrt{(r_{p\alpha}-(m_2'')^2)}} = A_9 \tag{37}$$

m_2'' can be obtained, and then

$$P/P_E = A_9^2/\pi^2,\quad s_9 = \frac{A_9 m_1}{\sqrt{(r_{p\alpha}-(m_2'')^2)}+\sqrt{\left(\frac{4}{z_1^2(1-p)}\right)}},\quad c_9 = m_2''/m_1 \tag{38}$$

Case 10. $(m_1 > \alpha, m_2 > \alpha, 0.5(m_2 - 1) < p < 1)$, m_1 in high position. This is the high position of either Case 7 or Case 8.

$$A''B''/\lambda = (1-p)^{3/2}\left[\ln\frac{r+1}{r-1}+\frac{2r}{r^2-1}\right]+2\sin^{-1}\frac{\alpha}{\sqrt{(r_{p\alpha})}}+0.5(1-p)^{3/2}\left[\ln\frac{z_1+1}{z_1-1}+\frac{2z_1}{z_1^2-1}\right]$$
$$-0.5(1-p)^{3/2}\left[\ln\frac{z_2+1}{z_2-1}+\frac{2z_2}{z_2^2-1}\right]=A_{10} \tag{39}$$

m_2'' may be had from solving

$$\frac{m_1+m_2''}{\sqrt{(z_p)}} = A_{10} \tag{40}$$

and the stiffness and carry-over factors

$$P/P_E = A_{10}^2/\pi^2,\quad s_{10} = \frac{A_{10} m_1}{\sqrt{(z_p)}+\sqrt{\left(\frac{4}{z_1^2(1-p)}\right)}},\quad c_{10} = m_2''/m_1 \tag{41}$$

follow